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Four-dimensional QCD and fiberwise duality

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Abstract: We transform, by means of a fiberwise duality, the partition function of QCD on a product of two two-tori, into a four-dimensional σ-model, whose target space is the cotangent space of unitary connections on the fiber torus fiberwise.

Keywords: Duality in Gauge Field Theories, Gauge Symmetry.

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Contents

1. Introduction

The prominent role that integrable systems (Hitchin systems [1]), defined over the cotangent bundle of unitary connections on a Riemann surface, play in the solution [2] of the low energy limit of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and certain string theories [3], has stimulated efforts to extend to other non-supersymmetric theories the $\mathcal{N} = 2$ techniques [4].

Yet, the field theoretical origin, in the four-dimensional gauge theory, of the Hitchin systems, remains a mystery.

This paper was inspired by a paper of Bershadsky, Johansen, Sadov and Vafa [5], about topologically reducing four-dimensional supersymmetric or non-supersymmetric gauge theories to two dimensions.

In the $\mathcal{N} = 4$ four-dimensional case, it was found that the topologically reduced two-dimensional σ-model had, as target space, a certain Hitchin fibration of the cotangent bundle of unitary connections.

In this paper, we show that the same is true for four-dimensional YM- theories, without any dimensional reduction, provided the entire infinite-dimensional cotangent bundle is taken as target and the σ-model is allowed to be four-dimensional and non-local.

More precisely, we consider a YM theory on a four-torus, that we think as a two-torus fibered on a two-torus, that is, as an elliptic manifold, with gauge group $SU(N)$, without matter fields.

We show that the partition function of this theory may, in a natural way, be exactly represented as a non-linear and non-local four-dimensional σ-model, whose target space is the cotangent space of two-dimensional unitary connections, $T^*\mathcal{A}$, on the fiber torus fiberwise.

The change of variables that leads to the cotangent bundle was suggested to the author by the existence of a well known duality transformation [6, 7], acting on

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the function field space of the YM functional integral. This duality transformation maps the partition function of YM in four dimensions into the partition function of a theory that, expressed through the dual fields, has an action with the same local part, up to some boundary terms, but with the role of weak and strong coupling interchanged.

In addition, the dual theory receives non-local contributions, unless certain functional determinants, that appear in the course of performing the duality transformation, cancel each other [8]. Though this cancellation will not be investigated in this paper, since the only existence of the duality transformation is relevant to our main argument, it was observed in ref. [8] that this cancellation of determinants would be essentially equivalent to a property that in modern terms is called S-duality for the partition function of the pure $SU(N)$ gauge theory with a topological θ-term.

In the course of performing the duality transformation, in addition to the four-dimensional connection one-form A, the dual connection one-form A^D and the dual field strength two-form K are introduced as auxiliary fields. Then, after integrating over the A field and some field redefinition, one gets the Bianchi identity constraint for the K field, that is solved as $K = F(A^D)$. Now, our key point is that, instead of eliminating all the A and K fields and keeping only the A^D field, as in the usual definition of the duality transformation, it is more interesting to integrate out all the fields but the ones that carry only the indices that label tangent directions to the fiber torus of our elliptic fibration. We call this a partial or fiberwise duality transformation. Employing complex coordinates on the fiber torus, $(z, \bar{z})$, and on the base torus, $(u, \bar{u})$, these fields are: $\left(A_z, A_{\bar{z}}, A^D_z, A^D_{\bar{z}}\right)$ and $K_{z\bar{z}} = F_{z\bar{z}}(A^D)$.

Remarkably, this integration can be done explicitly, by means of a trick, integrating only Gaussian functionals. The resulting functional integral is then a four-dimensional non-linear and non-local σ-model, whose target space has coordinates $\left(A_z, A_{\bar{z}}, A^D_z, A^D_{\bar{z}}\right)$. Performing the shift $A^D_z = A_z + \Psi_z, A^D_{\bar{z}} = A_{\bar{z}} + \Psi_{\bar{z}}$ with Ψ a two-dimensional one-form fiberwise, the target space of the σ-model becomes the cotangent space of the unitary connections $T^*\mathcal{A}$ on the fiber torus fiberwise.

2. A preparatory trick

In this section we put the YM functional integral in a form suited for performing the fiberwise duality transformation. Incidentally, we also write a gauge-fixed formula for the partition function in terms of only two physical polarizations. The YM partition function is given by the formula:

$$Z = \int \exp\left[-\frac{1}{4g^2}\int \sum_{\mu\nu} Tr(F^2_{\mu\nu}) d^4x\right] DA \,,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \,. \qquad (2.1)$$

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Tangent directions to the fiber torus are labelled by $\|$, while directions tangent to the base by $\perp$. With this notation, introducing the auxiliary variable $E_\perp$, the partition function can be written as a Gaussian integral over $E_\perp$:

$$Z = \int \exp\left[-\frac{1}{2g^2}\sum_{\|\perp}\int Tr\left(F_\|^2 + E_\perp^2 - 2iE_\perp F_\perp + F_{\|\perp}^2\right) d^4x\right] DADE_\perp \,. \quad (2.2)$$

The integration over the components of the connection transverse to the fiber, $A_\perp$, is Gaussian and can be performed explicitly, giving as a result the effective action for $E_\perp$ and $A_\|$, that is our desired preparatory result:

$$\begin{aligned} Z = \int \exp&\left[-\frac{1}{2g^2}\int Tr(F_\|^2 + (\partial_\perp A_\|)^2 + E_\perp^2) d^4x\right] \times \\ &\times \exp\left[\frac{1}{2g^2}\int Tr\left[\left(\partial_{\bar u}(\partial_\| A_\| - E_\perp) - j_{\bar u}\right)\left(-\Delta_{A_\|} - i\, ad_{E_\perp}\right)^{-1} \times \right.\right. \\ &\qquad \left.\left. \times \left(\partial_u(\partial_\| A_\| + E_\perp) - j_u\right)\right]\right] \times \\ &\times Det[-\Delta_{A_\|} - i\, ad_{E_\perp}]^{-1} DA_\| DE_\perp \,, \end{aligned} \quad (2.3)$$

where

$$j_u = i[\partial_u A_\|, A_\|]\,, \qquad j_{\bar u} = i[\partial_{\bar u} A_\|, A_\|] \quad (2.4)$$

and the sum over the appropriate space-time indices is understood. $\Delta_{A_\|}$ is the two-dimensional scalar Laplacian in the background of the connection $A_\|$ and $ad_{E_\perp}$ the adjoint action of the Lie algebra valued field $E_\perp$. Incidentally $E_\perp$ can be integrated, with the help of a convenient choice of the gauge, that has also the advantage of eliminating some of the non-local terms in the functional integral. The gauge choice is:

$$\partial_\| A_\| + E_\perp = 0\,. \quad (2.5)$$

Inserting this gauge condition and the corresponding Faddeev-Popov determinant, we get:

$$\begin{aligned} Z = \int \exp&\left[-\frac{1}{2g^2}\int Tr\left(F_\|^2 + \left(\partial_\perp A_\|\right)^2 + \left(\partial_\| A_\|\right)^2\right)\right] \times \\ &\times \exp\left[-\frac{1}{2g^2}\int Tr\left[(2\partial_{\bar u}\partial_\| A_\| - j_{\bar u})(-\Delta_{A_\|} + i\, ad_{\partial_\| A_\|})^{-1} j_u\right]\right] \times \\ &\times Det\left[-\Delta_{A_\|} + i\, ad_{\partial_\| A_\|}\right]^{-1} \Delta_{FP} DA_\| \,. \end{aligned} \quad (2.6)$$

The partition function is now a σ-model with only two polarizations.

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3. The duality transformation

We remind the reader the well known four-dimensional duality transformation, that is the starting point of our argument, following the lines of refs. [6, 7, 8]. The Euclidean partition function of four-dimensional YM with a θ-term is defined by the formula:

$$\begin{aligned} Z &= \int \exp\left[-\frac{1}{4g^2}\int\left(F^2 - i\tilde{\theta}F\tilde{F}\right)\right] DA\,, \\ \tilde{\theta} &= \frac{g^2}{8\pi^2}\theta\,, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right]\,, \end{aligned} \tag{3.1}$$

where the dual field strength is:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{3.2}$$

and the trace and sum over the Euclidean indices are understood. Employing the auxiliary variable K, the partition function can be written as a Gaussian integral over K:

$$Z = \int \exp\left[-\int \frac{g^2}{4\left(1+\tilde{\theta}^2\right)}\left(K^2 + i\tilde{\theta}K\tilde{K}\right) + \frac{i}{2}\tilde{K}F(A)\right] DADK\,. \tag{3.3}$$

After some manipulation, the following identity was found in ref. [7]:

$$\begin{aligned} Z = \int \exp&\left[-\int \frac{g^2}{4\left(1+\tilde{\theta}^2\right)}\left(K^2 + i\tilde{\theta}K\tilde{K}\right) + \frac{i}{2}\tilde{K}F(A^D) - \frac{i}{2}\xi_\Sigma \left(ad_{\tilde{K}}\right)^{-1}\xi_\Sigma\right] \times \\ &\times \left|\mathrm{Det}(ad_{\tilde{K}})\right|^{\frac{1}{2}} \delta\left(d^*_{A^D}\tilde{K}\right) DA^D DK\,, \end{aligned} \tag{3.4}$$

where ξ_Σ is a functional of K that depends only on the boundary values of K [7]. The delta-functional constraint is the Bianchi identity, that is solved by:

$$K = F(A^D) \tag{3.5}$$

up to some field-strength copies [9, 10]. In this way we get, for the final form of the partition function as a functional integral over the dual variables A^D [8]:

$$\begin{aligned} Z = \int \exp&\left[-\int \frac{g^2}{4\left(1+\tilde{\theta}^2\right)}\left(F(A^D)^2 + i\tilde{\theta}\tilde{F}(A^D)F(A^D)\right) + \right. \\ &\left. + \frac{i}{2}\tilde{F}(A^D)F(A^D) - \frac{i}{2}\xi_\Sigma\left(ad_{\tilde{F}(A^D)}\right)^{-1}\xi_\Sigma\right] \times \\ &\times \left|\mathrm{Det}\left(ad_{\tilde{F}(A^D)}\right)\right|^{\frac{1}{2}} \left|\mathrm{Det}\left(d^*_{A^D}\right)\right|^{-1} DA^D\,. \end{aligned} \tag{3.6}$$

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Instead of using as fundamental fields, through which to express the QCD functional integral, the usual gauge potential, A, or its dual, A^D, we choose $A_z, A_{\bar{z}}, A^D_z, A^D_{\bar{z}}$. Therefore we imagine of having integrated out all the other fields. The main object that we want to compute is the corresponding effective action, Γ, in terms of these fields:

$$Z = \int \exp\left[-\Gamma\left(A_z, A_{\bar{z}}, A^D_z, A^D_{\bar{z}}\right)\right] DA_z DA_{\bar{z}} DA^D_z DA^D_{\bar{z}}\,. \tag{3.7}$$

Instead of computing it directly, which is rather cumbersome and perhaps impossible using only Gaussian integrations, we make use of the following trick. After the change of variables necessary to perform the duality transformation, the Bianchi identity constraint is implemented in the functional integral via a delta-function, that is obtained integrating over the original vector potential:

$$\delta(d^*_{A^D}\tilde{K}) = \int \exp\left[i\int\left(\partial_\mu \tilde{K}_{\mu\nu} - i\left[\tilde{K}_{\mu\nu}, A^D_\mu\right]\right) A_\nu\right] DA_\nu\,. \tag{3.8}$$

At this stage, inserting in the functional integral the constraint:

$$K_{z\bar{z}} = F_{z\bar{z}}(A^D) \tag{3.9}$$

is equivalent to multiply the partition function by an harmless infinite factor, since eq. (3.9) is implied by the Bianchi identity constraint. If we want to introduce, instead, this constraint, before having performed the change of variables needed to define A^D, we must compensate the change of variables from $K_{z\bar{z}}$ to A^D with a Faddeev-Popov determinant, Δ_D. Therefore we get:

$$\begin{aligned} Z = \int \exp&\left[-\int \frac{g^2}{4\left(1+\tilde{\theta}^2\right)}\left(K^2 + i\tilde{\theta}K\tilde{K}\right) + \frac{i}{2}\tilde{K}F(A)\right] \times \\ &\times \delta\left(K_{z\bar{z}} - F_{z\bar{z}}(A^D)\right)\Delta_D DA^D_z DA^D_{\bar{z}} DADK\,. \end{aligned} \tag{3.10}$$

4. The fiberwise duality transformation

We are now ready to change variables in order to embed $T^*\mathcal{A}$ into the QCD functional integral. We simply set

$$E_\perp = \epsilon_{\perp\|}F_\|(A_D) \tag{4.1}$$

into eq. (2.3), where $\epsilon_{\perp\|}$ is the rank four normalized antisymmetric tensor and A_D is a connection form on the fiber torus T^2_f fiberwise. We call this change of variables a fiberwise duality transformation because it has the structure of a duality transformation restricted to the fiber. As in eq. (3.10), we introduce the resolution of the identity by means of the classical Faddeev-Popov trick:

$$1 = \Delta_D(E)\int \delta\left[E - F(A_D)\right] DA_D\,, \tag{4.2}$$

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where, in this section, $F(A_D)$ is the curvature two-form of the dual connection one-form A_D fiberwise. From now on we will refer to $A_{\|}$ as to A. It is convenient to decompose A_D into A and an arbitrary one-form Ψ:

$$A_D = A + \Psi\,. \tag{4.3}$$

Correspondingly, the Faddeev-Popov trick becomes, after shifting the A_D integration by A:

$$1 = \Delta_D(E, A) \int \delta\left[E - F(A + \Psi)\right] D\Psi\,. \tag{4.4}$$

We may consider the fields (A, Ψ) as the coordinates of $T^*\mathcal{A}$, the cotangent space of unitary connections, A, on the fiber torus. Of course (A, Ψ) are four-dimensional fields as functions on space-time, but they belong to $T^*\mathcal{A}$ fiberwise. This gives, after adding the gauge-fixing condition, the desired embedding of $T^*\mathcal{A}/G$, fiberwise in the functional integral:

$$\begin{aligned}
Z = \int \exp\left[-\frac{N}{2\lambda}\int Tr\left(F_A^2 + (d^*A)^2 + (\partial_\perp A)^2\right) d^4x\right] \times \\
\times \exp\left[\frac{N}{2\lambda}\int Tr\Big[(\partial_{\bar u}(\partial_{\|}A_{\|} - F_{A+\Psi}) - j_{\bar u})(-\Delta_{A_{\|}} - i\, ad_{F_{A+\Psi}})^{-1} \times \right.\\
\left.\times\, (\partial_u(\partial_{\|}A_{\|} + F_{A+\Psi}) - j_u)\Big]\right] \times \\
\times \operatorname{Det}\left[-\Delta_A - i\, ad_{F_{A+\Psi}}\right]^{-1} \Delta_D(A, \Psi)\Delta_{FP} \times \\
\times\, \delta\left[d^*A + F_{A+\Psi}\right] DAD\Psi\,.
\end{aligned} \tag{4.5}$$

References

[1] N. Hitchin, *Duke Math. Jour.* **54,1** (1987) 91.

[2] N. Seiberg and E. Witten, *Nucl. Phys.* **B 426** (1994) 19, erratum *ibid.* **B 430** (1994) 485 [hep-th/9407087].

[3] R. Dijkgraaf, *Les Houches lectures on fields, strings and duality*, hep-th/9703136 and references therein.

[4] L. Alvarez-Gaumé, J. Dister, C. Kounnas and M. Marino, *Int. J. Mod. Phys.* **A 11** (1996) 4745 [hep-th/9604004].

[5] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, *Nucl. Phys.* **B 448** (1995) 166 [hep-th/9501096].

[6] M.B. Halpern, *Phys. Rev.* **D 16** (1977) 1798;
M.B. Halpern, *Phys. Rev.* **D 19** (1979) 517.

[7] W. Mecklenburg and L. Mizrachi, *Phys. Rev.* **D 27** (1983) 1922.

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[8] M. Bochicchio, *Some comments on S-duality in four-dimensional QCD*, hep-th/9810012.

[9] M.B. Halpern, *Nucl. Phys.* **B 139** (1978) 477.

[10] P. Majumdar and H.S. Sharatchandra, *Duality transformations for* $3+1$ *dimensional Yang-Mills theory* , hep-th/9805102 and references therein.

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